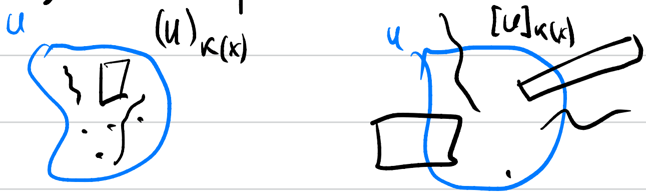
Descriptive Set Theory
Lecture 6

The hyperspace of compact subsets. Let X be a top. space and let $\mathcal{K}(X)$ denote the set of all of the compact subsets of X, including $\emptyset$. For general $\mathcal{A} \subseteq \mathcal{P}(X)$, we define sets of the following form:

$$(U)_{\mathcal{A}} := \{A \in \mathcal{A} : A \subseteq U\}$$
$$[U]_{\mathcal{A}} := \{A \in \mathcal{A} : A \cap U \neq \emptyset\},$$

for an open subset $U \subseteq X$. The Vietoris top on $\mathcal{K}(X)$ is the one generated by the sets $(U)_{\mathcal{K}(X)}$ and $[U]_{\mathcal{K}(X)}$, where U ranges over open subsets of X.

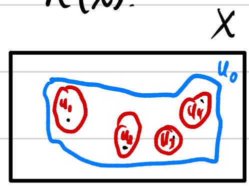


Thus finite intersections of sets of the form $(U)_{\mathcal{K}(X)}$ or $[U]_{\mathcal{K}(X)}$ form a basis for the top. on $\mathcal{K}(X)$, namely, sets of the form:

$$\langle U_0; U_1, U_2, \dots, U_n \rangle := (U_0)_{\mathcal{K}(X)} \cap [U_1]_{\mathcal{K}(X)} \cap \dots \cap [U_n]_{\mathcal{K}(X)},$$

where $n \geq 0$, we may assume WLOG that $U_1, \dots, U_n \subseteq U_0$ (by taking $U_i \cap U_0$ instead of U_i).

Prop. If X is a separable top. space, then so is $\mathcal{K}(X)$.



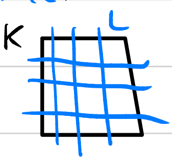
Proof. Let $\mathcal{D} \subseteq X$ be ctbl dense.

Then $\mathcal{F} := \mathcal{P}_{fin}(\mathcal{D}) := \{\text{finite subsets of } \mathcal{D}\}$ is ctbl and we show that it's dense in $\mathcal{K}(X)$.

Let $\langle U_0; U_1, \dots, U_n \rangle$ be a basic open set in $\mathcal{K}(X)$, where we assume $U_i \subseteq U_0$ $\forall i = 1, \dots, n$. By the density of $\mathcal{D}$, $\exists\, d_1, d_2, \dots, d_n \in \mathcal{D}$ s.t. $d_i \in U_i$, so the compact set $\{d_1, d_2, \dots, d_n\} \in \langle U_0; U_1, \dots, U_n \rangle$. $\square$

Note: The set $\emptyset \in \mathcal{K}(X)$ is an isolated point, i.e. $\{\emptyset\}$ is open in $\mathcal{K}(X)$; indeed, $(\emptyset)_{\mathcal{K}(X)} = \{\emptyset\}$.

When X is metrizable, then we can equip $\mathcal{K}(X)$ with the **Hausdorff metric**. Let d be a metric for X. Let $K, L \in \mathcal{K}(X)$:



$$d_H(K, L) = \inf_{r>0} \left[K \subseteq B(L, r) \text{ and } L \subseteq B(K, r) \right],$$

where $B(K, r) := \{x \in X : d(K, x) < r\}$.

It's convenient to also define the non symmetric version:

$$\vec{d}_H(K, L) := \inf_{r>0} \left[K \subseteq B(L, r) \right].$$

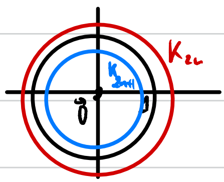
Thus, $d_H(K, L) = \min\{\vec{d}_H(K, L), \vec{d}_H(L, K)\}$.

Prop. Hausdorff metric is compatible with the Vietoris topology.

Proof. Left as an exercise. $\boxtimes$

We now study convergence in $K(X)$ having the goal of proving that if (X,d) is complete then so is $(K(X), d_H)$.

Example. Let $(X,d) := (\mathbb{R}^2, \text{Euclidean})$ and $K_n = \overline{B}(\vec{0}, 1 + \frac{(-1)^n}{n})$.



Then $d_H(K_n, \overline{B}(\vec{0},1)) = \frac{1}{n}$ $\forall n$ so $K_n \to \overline{B}(\vec{0},1)$ in d_H.

Given a d_H-Cauchy sequence (K_n), we need to define a potential limit K for it. The following constructions are some possibilities:

For any sequence K_n of subsets of X,
the topological upper limit of (K_n) is

$$\overline{T\lim_n} K_n := \{x \in X : \text{every open } U \ni x \text{ meets } \infty \text{ many of the } K_n\},$$

and the top. lower limit of (K_n) is

$$\underline{\mathrm{Tlim}}_n K_n := \{x \in X : \text{every open } U \ni x \text{ meets all but fin. many } K_n\}.$$

Note that both $\underline{\mathrm{Tlim}}$ and $\overline{\mathrm{Tlim}}$ are closed regardless of (K_n).

Note that $\overline{\mathrm{Tlim}}_n K_n = \bigcap_{i=1}^{\infty} \overline{\bigcup_{j \geq i} K_j}$.

If X is 1^{st}-ctbl (e.g. metrizable), then we also have sequential representation of these top. limits:

$$\overline{\mathrm{Tlim}}_n K_n = \{x \in X : \exists (x_{n_k}),\ x_{n_k} \in K_{n_k},\ (x_{n_k})_k \to x\}.$$

$$\underline{\mathrm{Tlim}}_n K_n = \{x \in X : \exists (x_n),\ \forall^\infty_n\ x_n \in K_n,\ (x_n) \to x\}.$$

Clearly, $\underline{\mathrm{Tlim}}_n K_n \subseteq \overline{\mathrm{Tlim}}_n K_n$ and when they are equal, we just call it the "top. limit and denote it by $\mathrm{Tlim}_n K_n$.

Prop. If $K_n, K \in \mathcal{K}(X)$ and $K_n \to_{d_H} K$, then $\mathrm{Tlim}_n K_n = K$.

Proof. Exercise. $\boxtimes$

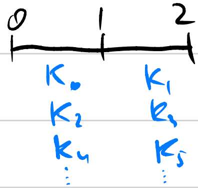
Examples. (a) $X := \mathbb{R}$ and $K_n := [-n, n]$. Then $\mathrm{Tlim}_n K_n = \mathbb{R}$.

Thus, the top. lim. of compact sets need not be compact.

(b) Let $X := \mathbb{R}$ and $K_n := [0,1] \cup [n, n+1]$, $n \geq 1$.

$\underline{\lim}_n K_n = [0,1] = \overline{\lim}_n K_n$ so $\lim_n K_n = [0,1]$, yet $d_H(K_n, K_m) \geq 1$ $\forall n \neq m$, so (K_n) doesn't converge in Hausdorff metric.

(c) Let $X := \mathbb{R}$ and $K_n := \begin{cases} [0,1] & \text{if } n \text{ is even} \\ [1,2] & \text{if } n \text{ is odd} \end{cases}$.



$\underline{\lim}_n K_n = \{1\}$ while $\overline{\lim}_n K_n = [0,2]$.

Theorem. If (X,d) is complete, then $(\mathcal{K}(X), d_H)$ is complete.

Proof. Let $(K_n) \subseteq \mathcal{K}(X)$ be a d_H-Cauchy sequence.

Let $K := \overline{\lim}_n K_n = \bigcap_n \overline{\bigcup_{m \geq n} K_m}$

$$= \{x \in X : \exists (x_{n_k}) \to x, x_{n_k} \in K_{n_k}\}.$$

Claim. K is compact.

Proof. K is closed, so it remains to show that it's totally bounded. Let $\varepsilon > 0$. Let N be large enough so that $\forall n, m \geq N$, $d_H(K_n, K_m) < \frac{\varepsilon}{4}$.

Let F be a finite $\frac{\varepsilon}{4}$-net for K_N. Then $K_N \subseteq B(F, \frac{\varepsilon}{4})$ and $\forall n \geq N$, $K_n \subseteq B(K_N, \frac{\varepsilon}{4})$, so $\forall n$, $K_n \subseteq B(F, \frac{\varepsilon}{4} + \frac{\varepsilon}{4})$. Thus, $\overline{\bigcup_{n \geq N} K_n} \subseteq B(F, \varepsilon)$. Hence, $K \subseteq B(F, \varepsilon)$, so F is an ε-net for K. $\square$

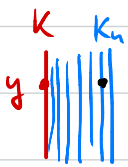
Next we show that $d_H(K_n, K) \to 0$. Let $\varepsilon > 0$ and let N be large enough so that $d_H(K_n, K_m) < \frac{\varepsilon}{2}$ $\forall n, m \geq N$.

Claim 2. $\vec{d}_H(K, K_n) < \varepsilon$ $\forall n \geq N$.

Proof. $K \subseteq \overline{\bigcup_{m \geq n} K_m} \subseteq \overline{B(K_n, \frac{\varepsilon}{2})} \subseteq B(K_n, \varepsilon)$. $\square$

Claim 3. $\vec{d}_H(K_n, K) < \varepsilon$ $\forall n \geq N$.

Proof. We need to show that $K_n \subseteq B(K, \varepsilon)$, i.e. fixing $x \in K_n$, we need to find $y \in K$ s.t. $d(x,y) < \varepsilon$.



Let (K_{n_i}) be a subsequence, starting with $n_0 := n$, s.t. $d_H(K_{n_i}, K_{n_{i+1}}) < \frac{\varepsilon}{2} \cdot 2^{-i}$. Then $\exists (x_i)$, starting $x_0 := x$, s.t. $x_i \in K_{n_i}$ and (x_i) is d-Cauchy (do this inductively, using $d_H(K_{n_i}, K_{n_j}) < \frac{\varepsilon}{2} \cdot 2^{-i}$ $\forall j \geq i$). Then by the completeness of d, $\exists y \in X$ s.t. $x_i \to y$

as $i \to \infty$. But then by the sequential characterization of $\overline{\mathrm{T}}\lim_n K_n$, $y \in K$. $\square$ $\boxtimes$

Corollary. If X is Polish, then so is $K(X)$.

Corollary. If X is compact metrizable, then so is $K(X)$.

Proof. Because we already know that d_H is complete, it remains to show that $K(X)$ is totally bounded. Fix $\varepsilon > 0$. Let F be a finite ε-net for X. Then $\mathcal{P}(F)$ is finite and we check that $\mathcal{P}(F)$ is an ε-net for $K(X)$. Indeed, let $K \in K(X)$ and take $F_K := \{x \in F : B(x,\varepsilon) \cap K \neq \emptyset\}$. Then because $X = \bigcup_{x \in F} B(x,\varepsilon)$, $K \subseteq \bigcup_{x \in F_K} B(x,\varepsilon) = B(F_K, \varepsilon)$. Also, $F_K \subseteq B(K,\varepsilon)$ by definition, so $d_H(F_K, K) < \varepsilon$. $\square$